UNIT-III

Time domain analysis: **Introduction, standard test signals- time response specifications-steady state error constants**

TIME RESPONSE

The time response of the system is the output of the closed loop system as a function of time. It is denoted by c(t). The time response can be obtained by solving the differential equation governing the system. Alternatively, the response c(t) can be obtained from the transfer function of the system and the input to the system.

The closed loop transfer function, $$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)} = M(s) \qquad .....(2.1)$$

The Output or Response in s-domain, C(s) is given by the product of the transfer function and the input, R(s). On taking inverse Laplace transform of this product the time domain response, c(t) can be obtained.

Response in s-domain, $$C(s) = R(s)\,M(s) \qquad .....(2.2)$$

Response in time domain, $$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\{R(s) \times M(s)\} \qquad .....(2.3)$$

where, $$M(s) = \frac{G(s)}{1+G(s)H(s)}$$

The time response of a control system consists of two parts : *the transient and the steady state response.* The transient response is the response of the system when the input changes from one state to another. The steady state response is the response as time, **t** approaches infinity.

TEST SIGNALS

The knowledge of input signal is required to predict the response of a system.

The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity and a constant acceleration. Hence test signals which resembles these characteristics are used as input signals to predict the performance of the system. The commonly used test input signals are impulse, step, ramp, acceleration and sinusoidal signals.

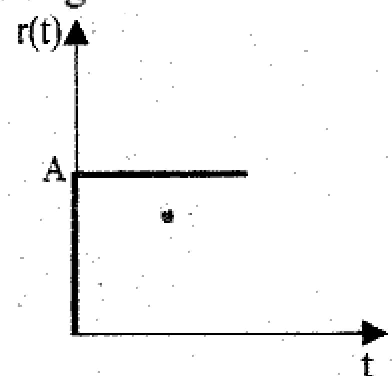
The standard test signals are,

1. a) Step signal
   b) Unit step signal
2. a) Ramp signal
   b) Unit ramp signal
3. a) Parabolic signal
   b) Unit parabolic signal
4. Impulse signal
5. Sinusoidal signal.

Since the test signals are simple functions for time, they can be easily generated in laboratories.

STEP SIGNAL

The step signal is a signal whose value changes from zero to A at $t = 0$ and remains constant at A for $t > 0$.



r(t)=A u(t)

u(t)=1 for t $\geq$ 0

=0 for t < 0

Laplace transform of step signal is R(S)= A/s

A special case of step signal is unit step in which **A** is unity.

RAMP SIGNAL

The ramp signal is a signal whose value increases linearly with time from an initial value of zero at t = 0.

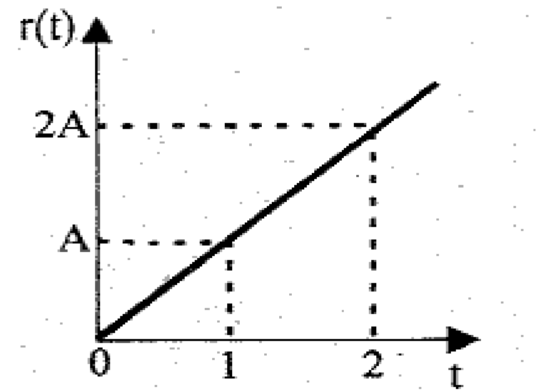


Fig 2.3 : Ramp signal.

The mathematical representation of the ramp signal is,

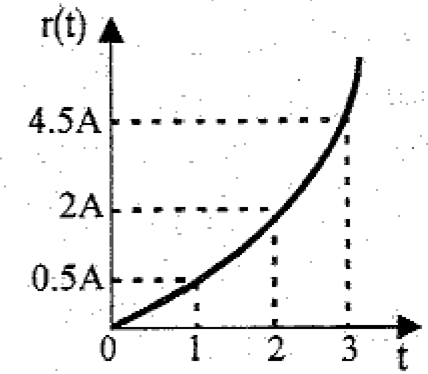
$$r(t) = A\,t \quad ; \quad t \geq 0$$
$$= 0 \quad ; \quad t < 0$$

Laplace transform of ramp signal is R(S)= A/S^2

A special case of ramp signal is unit ramp signal in which the value of **A** is unity.

PARABOLIC SIGNAL

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at $t = 0$. The sketch of the signal with respect to time resembles a parabola.



The mathematical representation of the parabolic signal is,

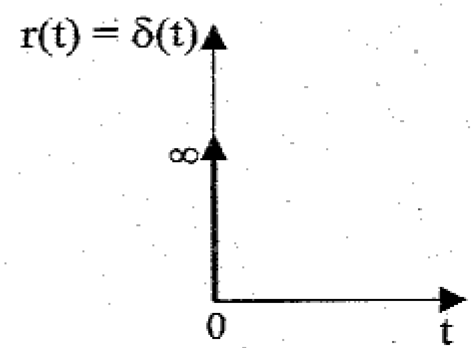
$$r(t) = \frac{At^2}{2} \quad ; \quad t \geq 0$$
$$= 0 \quad ; \quad t < 0$$

A special case of parabolic signal is unit parabolic signal in which A is unity.

Integral of step signal is ramp signal. Integral of ramp signal is parabolic signal.

IMPULSE SIGNAL

A signal of very large magnitude which is available for very short duration is called ***impulse signal.*** Ideal impulse signal is a signal with infinite magnitude and zero duration



$Where, \omega = 2\pi f$

The impulse signal is denoted by δ(t) and nathematically it is expressed as,

$$\delta(t) = \infty;\ t = 0$$

$$= 0;\ t \neq 0$$

An impulse function is derivative of step function

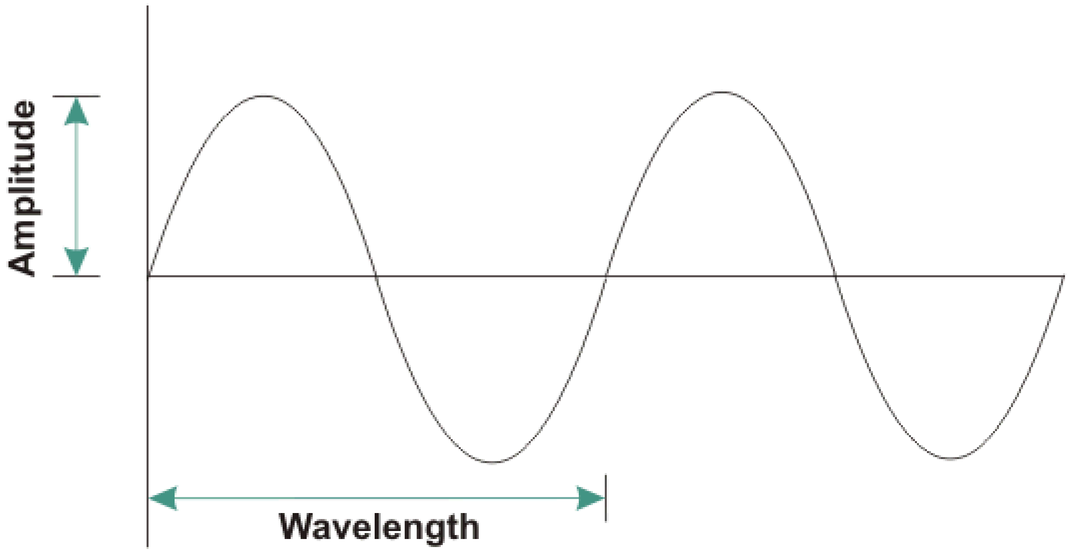
Sinusoidal Wave Signal

Sine Wave or Sinusoidal Wave Signal is a special type of signal. It is given by the function

$$f(t) = \sin(\omega t) \quad or \quad f(t) = A\sin(\omega t + \phi)$$
$$Where, \omega = 2\pi f$$

When Sine wave starts from zero and covers positive values, reaches zero; and again covers negative values, reaches zero, it is said to have completed one cycle or single cycle.

The upper part of sine wave is called positive cycle and the lower part is called negative cycle in a single cycle.



The Maximum value of the Sinusoidal Signal is also called its amplitude (A). Here ω is called Angular Frequency of Signal and f is the Frequency of Signal. Ø Is called Phase difference.

Sinusoidal signals are important in both electrical and electronic engineering domains.

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	$\frac{A}{s}$
Unit step	1	$\frac{1}{s}$
Ramp	At	$\frac{A}{s^2}$
Unit ramp	t	$\frac{1}{s^2}$
Parabolic	$\frac{At^2}{2}$	$\frac{A}{s^3}$
Unit parabolic	$\frac{t^2}{2}$	$\frac{1}{s^3}$
Impulse	$\delta(t)$	1

TIME DOMAIN SPECIFICATIONS

The desired performance characteristics of control systems are specified in terms of time domain specifications.

The desired performance characteristics of a system of any order may be specified in terms of the transient response to a unit step input signal.

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications.

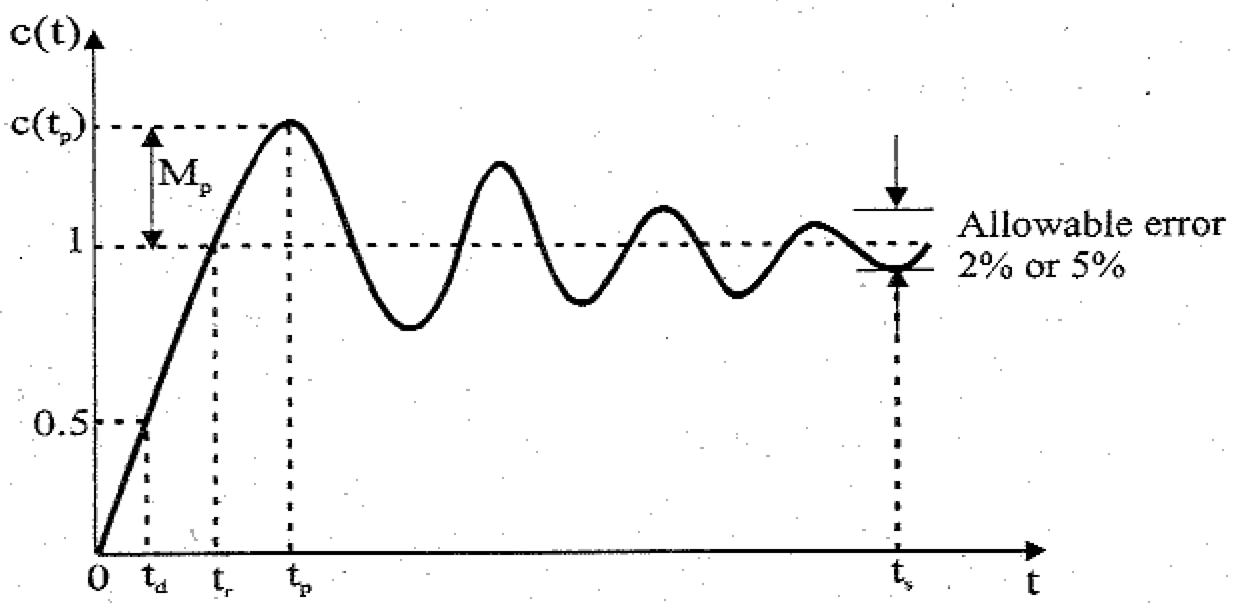
1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s

The time domain specifications are defined as follows.

1. DELAY TIME (t_d) : It is the time taken for response to reach 50% of the final value, for the very first time.

2. RISE TIME (t_r) : It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

3. PEAK TIME (t_p) : It is the time taken for the response to reach the peak value the very first time.



overshoot, M_p.

4. PEAK OVERSHOOT (M_p) : It is defined as the ratio of the maximum peak value to the final value where the maximum peak value is measured from final value.

Let, $c(\infty) =$ Final value of c(t).

$c(t_p) =$ Maximum value of c(t).

$$\text{Now, Peak overshoot, } M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

$$\text{\% Peak overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

5. SETTLING TIME (t_s) : It is defined as the time taken by the response to reach and stay within a specified error. It is usually expressed as % of final value. The usual tolerable error is 2 % or 5% of the final value.

EXPRESSIONS FOR TIME DOMAIN SPECIFICATIONS

Rise time (t_r)

The unit step response of under damped second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$

$$\therefore c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 1$$

$$\therefore \frac{-e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

Since $-e^{-\zeta\omega_n t_r} \neq 0$, the term, $\sin(\omega_d t_r + \theta) = 0$

When, $\phi = 0, \pi, 2\pi, 3\pi \ldots$, $\sin \phi = 0$

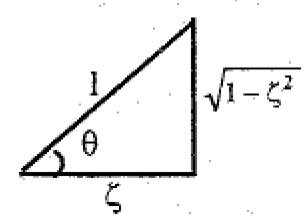
$$\therefore \omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$\therefore \text{Rise Time, } t_r = \frac{\pi - \theta}{\omega_d}$$

On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$



Here, $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$; Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \text{Rise time, } t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \text{ in sec}$$

θ or $\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ should be measured in radians.

Peak time (t_p)

To find the expression for peak time, t_p, differentiate c(t) with respect to t and equate to 0.

$$\text{i.e., } \frac{d}{dt}c(t)\Big|_{t=t_p} = 0$$

The unit step response of under damped second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

Differentiating c(t) with respect to **t**.

$$\frac{d}{dt}c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(-\zeta\omega_n)\sin(\omega_d t + \theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\cos(\omega_d t + \theta)\omega_d$$

Put, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \frac{d}{dt}c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\zeta\omega_n)\sin(\omega_d t+\theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_d t+\theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta\sin(\omega_d t+\theta) - \sqrt{1-\zeta^2}\;\cos(\omega_d t+\theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\cos\theta\sin(\omega_d t+\theta) - \sin\theta\cos(\omega_d t+\theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\sin(\omega_d t+\theta)\cos\theta - \cos(\omega_d t+\theta)\sin\theta\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\sin((\omega_d t+\theta)-\theta)\right] = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t)$$

at $t = t_p$, $\frac{d}{dt}c(t) = 0$

$$\therefore \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p}\sin(\omega_d t_p) = 0$$

Since, $e^{-\zeta\omega_n t_p} \neq 0$, the term, $\sin(\omega_d t_p) = 0$

When $\phi = 0, \pi, 2\pi, 3\pi$, $\sin\phi = 0$

$\therefore \; \omega_d t_p = \pi$

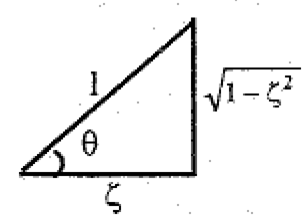
$$\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_d}$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$



Peak overshoot (M_p)

$$\%\text{Peak overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

where, $c(t_p)$ = Peak response at $t = t_p$.
$c(\infty)$ = Final steady state value.

The unit step response of second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$$\text{At } t=\infty, \quad c(t) = c(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) = 1 - 0 = 1$$

$$\text{At } t = t_p, \quad c(t) = c(t_p) = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta)$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \theta\right)$$

$$t_p = \frac{\pi}{\omega_d}$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \sin(\pi + \theta)}{\sqrt{1-\zeta^2}}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\theta$$

$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sqrt{1-\zeta^2} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

> ***Note :*** *On constructing right angle triangle with* ζ *and* $\sqrt{1-\zeta^2}$, *we get*
>
> $\sin\theta = \sqrt{1-\zeta^2}$
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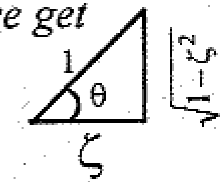
> 
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$$\text{Percentage Peak Overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 = \frac{1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - 1}{1} \times 100$$

$$= e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\therefore \text{ Percentage Peak Overshoot, } \%M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

Settling time (t_s)

The response of second order system has two components. They are,

1. Decaying exponential component, $\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$.
2. Sinusoidal component, $\sin(\omega_d t + \theta)$.

In this the decaying exponential term dampens (or) reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component. The settling time can be found out by equating exponential component to percentage tolerance errors.

For 2 % tolerance error band, at $t = t_s$, $\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$

For least values of ζ, $e^{-\zeta\omega_n t_s} = 0.02$.

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.02) \quad \Rightarrow \quad -\zeta\omega_n t_s = -4 \quad \Rightarrow \quad t_s = \frac{4}{\zeta\omega_n}$$

For the second order system, the time constant, $T = \frac{1}{\zeta\omega_n}$

$$\therefore \text{ Settling time, } t_s = \frac{1}{\zeta\omega_n} = 4T \qquad \text{(for 2\% error)}$$

For 5% error, $e^{-\zeta\omega_n t_s} = 0.05$

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.05) \quad \Rightarrow \quad -\zeta\omega_n t_s = -3 \quad \Rightarrow \quad t_s = \frac{3}{\zeta\omega_n}$$

$$\therefore \text{ Settling time, } t_s = \frac{3}{\zeta\omega_n} = 3T \qquad \text{(for 5\% error)}$$

In general for a specified percentage error, Settling time can be evaluted using equation

$$\therefore \text{ Settling time, } t_s = \frac{ln(\%\text{ error})}{\zeta\omega_n} = \frac{ln(\%\text{ error})}{T}$$

1) Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the input is unit step.

SOLUTION

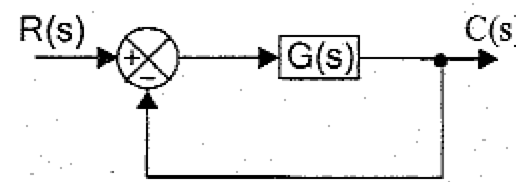


Fig 1 : Closed loop system.

The closed loop system is shown in fig 1.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{4}{s\,(s+5)}}{1+\frac{4}{s\,(s+5)}} = \frac{\frac{4}{s\,(s+5)}}{\frac{s\,(s+5)+4}{s\,(s+5)}} = \frac{4}{s\,(s+5)+4} = \frac{4}{s^2+5s+4} = \frac{4}{(s+4)\,(s+1)}$$

The response in s-domain, $C(s) = R(s)\,\frac{4}{(s+1)\,(s+4)}$

Since the input is unit step, $R(s) = \frac{1}{s}$; $\quad \therefore C(s) = \frac{4}{s\,(s+1)\,(s+4)}$

By partial fraction expansion, we can write,

$$C(s) = \frac{4}{s\,(s+1)\,(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = C(s) \times s\Big|_{s=0} = \frac{4}{(s+1)\,(s+4)}\Bigg|_{s=0} = \frac{4}{1\times 4} = 1$$

$$B = C(s) \times (s+1)\Big|_{s=-1} = \frac{4}{s\,(s+4)}\Bigg|_{s=-1} = \frac{4}{-1(-1+4)} = \frac{-4}{3}$$

$$C = C(s) \times (s+4)\Big|_{s=-4} = \frac{4}{s\,(s+1)}\Bigg|_{s=-4} = \frac{4}{-4(-4+1)} = \frac{1}{3}$$

The time domain response c(t) is obtained by taking inverse Laplace transform of C(s).

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+4}\right\}$

$$= 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t} = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$$

RESULT

Response of unity feedback system, $c(t) = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$

2)

A positional control system with velocity feedback is shown in fig 1. What is the response of the system for unit step input.

SOLUTION

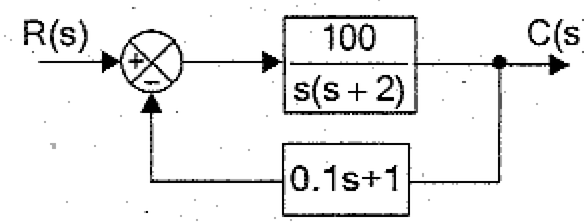


Fig 1 : Positional control system.

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$

Given that, $G(s) = \frac{100}{s(s+2)}$ and $H(s) = 0.1s + 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{100}{s\,(s+2)}}{1 + \left(\frac{100}{s\,(s+2)}\right)(0.1s+1)} = \frac{\frac{100}{s\,(s+2)}}{\frac{s(s+2) + 100\,(0.1s+1)}{s\,(s+2)}} = \frac{100}{s^2 + 2s + 10s + 100} = \frac{100}{s^2 + 12s + 100}$$

The response in s - domain, $C(s) = R(s)\ \dfrac{100}{s^2 + 12s + 100}$

Since input is unit step, $R(s) = \dfrac{1}{s}$

$$\therefore C(s) = \frac{1}{s}\ \frac{100}{s^2 + 12s + 100} = \frac{100}{s(s^2 + 12s + 100)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{100}{s\,(s^2 + 12s + 100)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 12s + 100}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times \left. s \right|_{s=0} = \left. \frac{100}{s^2 + 12s + 100} \right|_{s=0} = \frac{100}{100} = 1$$

$$\frac{100}{s\,(s^2 + 12s + 100)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 12s + 100}$$

$$100 = A(s^2 + 12s + 100) + (Bs + C)\,s$$

$$100 = As^2 + 12As + 100A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B$ $\therefore B = -A = -1$

On equating coefficients of s we get, $0 = 12A + C$ $\therefore C = -12A = -12$

$$\therefore C(s) = \frac{1}{s} + \frac{-s - 12}{s^2 + 12s + 100} = \frac{1}{s} - \frac{s + 12}{s^2 + 12s + 36 + 64} = \frac{1}{s} - \frac{s + 6 + 6}{(s+6)^2 + 8^2}$$

$$= \frac{1}{s} - \frac{s + 6}{(s+6)^2 + 8^2} - \frac{6}{(s+6)^2 + 8^2} = \frac{1}{s} - \frac{s + 6}{(s+6)^2 + 8^2} - \frac{6}{8}\ \frac{8}{(s+6)^2 + 8^2}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

$$\text{Time response, } c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\,\frac{8}{(s+6)^2+8^2}\right\}$$

$$= 1 - e^{-6t}\cos 8t - \frac{6}{8}\,e^{-6t}\sin 8t = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right]$$

3)

The response of a servomechanism is, $c(t) = 1 + 0.2\, e^{-60t} - 1.2\, e^{-10t}$ when subject to a unit step input. Obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

SOLUTION

Given that, $c(t) = 1 + 0.2\, e^{-60t} - 1.2\, e^{-10t}$

On taking Laplace transform of c(t) we get,

$$C(s) = \frac{1}{s} + 0.2\,\frac{1}{(s+60)} - 12\frac{1}{(s+10)} = \frac{(s+60)(s+10) + 0.2\,s\,(s+10) - 1.2\,s\,(s+60)}{s\,(s+60)\,(s+10)}$$

$$= \frac{s^2 + 70s + 600 + 0.2\,s^2 + 2s - 12s^2 - 72s}{s\,(s+60)\,(s+10)} = \frac{600}{s\,(s+60)\,(s+10)} = \frac{1}{s}\,\frac{600}{(s+60)\,(s+10)}$$

Since input is unit step, R(s) = 1/s.

$$\therefore\ C(s) = R(s)\ \frac{60}{(s+60)\ (s+10)} = R(s)\ \frac{600}{s^2+70s+600}$$

$\therefore$ The closed loop transfer function of the system, $\dfrac{C(s)}{R(s)} = \dfrac{600}{s^2+70s+600}$

The damping ratio and natural frequency of oscillation can be estimated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore\ \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{600}{s^2+70s+600}$$

On comparing we get,

$$\omega_n^2 = 600$$

$$\therefore\ \omega_n = \sqrt{600} = 24.49 \text{ rad / sec}$$

$$2\zeta\omega_n = 70$$

$$\therefore\ \zeta = \frac{70}{2\omega_n} = \frac{70}{2\times 24.49} = 1.43$$

RESULT

The closed loop transfer function of the system, $\dfrac{C(s)}{R(s)} = \dfrac{600}{s^2+70s+600}$

Natural frequency of oscillation, ω_n = 24.49 rad/sec

Damping ratio, ζ = 1.43

4)

The unity feedback system is characterized by an open loop transfer function G(s) =K/s (s +10). Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine peak overshoot and time at peak overshoot for a unit step input.

SOLUTION

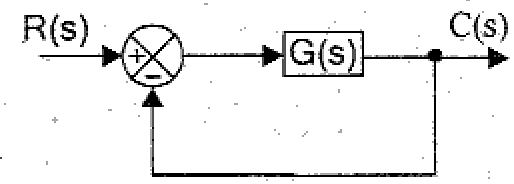


Fig 1 : Unity feedback system.

The unity feedback system is shown in fig 1.

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, G(s) = K/s (s +10)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s\,(s+10)+K} = \frac{K}{s^2+10s+K}$$

The value of K can be evaluated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K}{s^2+10s+K}$$

On comparing we get,

$\omega_n^2 = K$	$2\zeta\omega_n = 10$	$K = 100$
$\therefore \omega_n = \sqrt{K}$	Put $\zeta = 0.5$ and $\omega_n = \sqrt{K}$	$\omega_n = 10$ rad / sec
	$\therefore 2 \times 0.5 \times \sqrt{K} = 10$	
	$\sqrt{K} = 10$	

The value of gain, K=100.

Percentage peak overshoot, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 0.163 \times 100 = 16.3\%$$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.363$ sec

RESULT

The value of gain,	K	=	100
Percentage peak overshoot,	$\%M_p$	=	16.3%
Peak time,	t_p	=	0.363 sec.

5)

The open loop transfer function of a unity feedback system is given by $G(s) = K/s\,(sT + 1)$, where K and T are positive constant. By what factor should the amplifier gain K be reduced, so that the peak overshoot of unit step response of the system is reduced from 75% to 25%.

SOLUTION

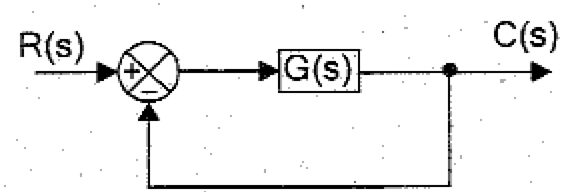


Fig 1 : Unity feedback system.

The unity feedback system is shown in fig 1.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, $G(s) = K/s\,(sT+1)$

$$\therefore \frac{C(s)}{R(s)} = \frac{K/s\,(sT+1)}{1+K/s(sT+1)} = \frac{K}{s\,(sT+1)+K} = \frac{K}{s^2T+s+K} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

Expression for ζ and ω_n can be obtained by comparing the transfer function with the standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

$$s^2 + \frac{1}{T}s + \frac{K}{T}$$

On comparing we get,

$$\omega_n^2 = K/T \qquad 2\zeta\omega_n = 1/T$$

$$\therefore \omega_n = \sqrt{K/T} \qquad \zeta = \frac{1}{2\omega_n T} = \frac{1}{2\sqrt{\frac{K}{T}}\,T} = \frac{1}{2\sqrt{KT}}$$

The peak overshoot, M_p is reduced by increasing the damping ratio ζ. The damping ratio ζ is increased by reducing the gain K.

When $M_p = 0.75$, Let $\zeta = \zeta_1$ and $K = K_1$

When $M_p = 0.25$, Let $\zeta = \zeta_2$ and $K = K_2$

Peak overshoot, $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Taking natural logarithm on both sides, $\ln M_p = \dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}$

On squaring we get, $(\ln M_p)^2 = \dfrac{\zeta^2\pi^2}{1-\zeta^2}$

On crossing multiplication we get,

$$(1-\zeta^2)\,(\ln M_p)^2 = \zeta^2\pi^2$$

$$(\ln M_p)^2 - \zeta^2(\ln M_p)^2 = \zeta^2\pi^2$$

$$(\ln M_p)^2 = \zeta^2\pi^2 + \zeta^2(\ln M_p)^2$$

$$(\ln M_p)^2 = \zeta^2 \left[\pi^2 + (\ln M_p)^2\right]$$

$$\therefore \zeta^2 = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2} \qquad \text{....(1)}$$

$$\text{But } \zeta = \frac{1}{2\sqrt{KT}}, \quad \therefore \zeta^2 = \frac{1}{4KT} \qquad \text{....(2)}$$

On equating, equation (1) & (2) we get,

$$\frac{1}{4KT} = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}$$

$$\frac{1}{K} = \frac{4T\,(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}$$

$$K = \frac{\pi^2 + (\ln M_p)^2}{4T\,(\ln M_p)^2}$$

$$\text{When, } K = K_1,\ M_p = 0.75, \quad \therefore K_1 = \frac{\pi^2 + (\ln 0.75)^2}{4T\,(\ln 0.75)^2} = \frac{9.952}{0.331T} = \frac{30.06}{T}$$

$$\text{When, } K = K_2,\ M_p = 0.25, \quad \therefore K_2 = \frac{\pi^2 + (\ln 0.25)^2}{4T\,(\ln 0.25)^2} = \frac{11.79}{7.68T} = \frac{1.53}{T}$$

$$\therefore \frac{K_1}{K_2} = \frac{(1/T)\ 30.06}{(1/T)\ 1.53} = 19.6$$

$$\text{When, } K = K_1,\ M_p = 0.75, \quad \therefore K_1 = \frac{\pi^2 + (\ln 0.75)^2}{4T\,(\ln 0.75)^2} = \frac{9.952}{0.331T} = \frac{30.06}{T}$$

$$\text{When, } K = K_2,\ M_p = 0.25, \quad \therefore K_2 = \frac{\pi^2 + (\ln 0.25)^2}{4T\,(\ln 0.25)^2} = \frac{11.79}{7.68T} = \frac{1.53}{T}$$

$$\therefore \frac{K_1}{K_2} = \frac{(1/T)\ 30.06}{(1/T)\ 1.53} = 19.6$$

$$K_1 = 19.6\, K_2 \qquad \text{(or)} \qquad K_2 = \frac{1}{19.6} K_1$$

To reduce peak overshoot from 0.75 to 0.25, K should be reduced by 19.6 times (approximately 20 times).

RESULT

The value of gain, K should be reduced approximately 20 times to reduce peak overshoot from 0.75 to 0.25.

6) A positional control system with velocity feedback is shown in fig 1. What is the response c(t) to the unit step input. Given that $\zeta = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.

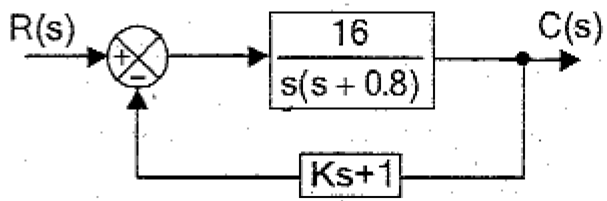


Fig 1

SOLUTION

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1 + G(s)\,H(s)}$

Given that G(s) = 16/s(s + 0.8) and H(s) = Ks + 1

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{16}{s\,(s+0.8)}}{1 + \dfrac{16}{s\,(s+0.8)}\,(Ks+1)} = \frac{16}{s\,(s+0.8) + 16(Ks+1)}$$

$$= \frac{16}{s^2 + 0.8s + 16Ks + 16} = \frac{16}{s^2 + (0.8 + 16K)\,s + 16}$$

The values of K and ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{16}{s^2 + (0.8 + 16K)\,s + 16}$$

On comparing we get.

$$\omega_n^2 = 16 \qquad\qquad 0.8 + 16K = 2\zeta\omega_n$$

$$\therefore \omega_n = 4 \text{ rad / sec} \qquad\qquad \therefore K = \frac{2\zeta\omega_n - 0.8}{16} = \frac{2 \times 0.5 \times 4 - 0.8}{16} = 0.2$$

$$\therefore \frac{C(s)}{R(s)} = \frac{16}{s^2 + (0.8 + 16 \times 0.2)\,s + 16} = \frac{16}{s^2 + 4s + 16}$$

Given that the damping ratio, $\zeta = 0.5$. Hence the system is underdamped and so the response of the system will have damped oscillations. The roots of characteristic polynomial will be complex conjugate.

The response in s - domain, $C(s) = R(s)\,\frac{16}{s^2 + 4s + 16}$

For unit step input, R(s) = 1/s.

$$\therefore\ C(s) = \frac{1}{s}\,\frac{16}{s^2 + 4s + 16} = \frac{16}{s\,(s^2 + 4s + 16)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{16}{s\,(s^2 + 4s + 16)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 16}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\Big|_{s=0} = \frac{16}{s^2 + 4s + 16}\Big|_{s=0} = \frac{16}{16} = 1$$

The residues B and C are evaluated by cross multiplying the following equation and equating the coefficients of like powers of s.

$$\frac{16}{s\,(s^2 + 4s + 16)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 16}$$

On cross multiplication we get, $16 = A\,(s^2 + 4s + 16) + (Bs + C)\,s$

$$16 = As^2 + 4As + 16A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B \quad \therefore \ B = -A = -1$

On equating the coefficients of s we get, $0 = 4A + C \ \therefore \ C = -4A = -4$

$$\therefore \ C(s) = \frac{1}{s} + \frac{-s-4}{s^2+4s+16} = \frac{1}{s} - \frac{s+4}{s^2+4s+4+12}$$

$$= \frac{1}{s} - \frac{s+2+2}{(s+2)^2+12} = \frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}} \frac{\sqrt{12}}{(s+2)^2+12}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}} \frac{\sqrt{12}}{(s+2)^2+12}\right\}$$

$$= 1 - e^{-2t} \cos\sqrt{12}\, t - \frac{2}{2\sqrt{3}} e^{-2t} \sin\sqrt{12}\, t$$

$$= 1 - e^{-2t}\left[\frac{1}{\sqrt{3}} \sin(\sqrt{12}\, t) + \cos(\sqrt{12}\, t)\right]$$

$$\left.\begin{array}{l}\text{Damped frequency}\\ \text{of oscillation}\end{array}\right\} \omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464 \text{ rad / sec}$$

$$\therefore \ \text{Rise time, } t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - 1047}{3.464} = 0.6046 \text{ sec}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907 \text{ sec}$$

$$\left.\begin{array}{l}\%\text{ Maximum}\\ \text{overshoot}\end{array}\right\} \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\times\pi}{\sqrt{1-0.5^2}}} \times 100 = 0.163 \times 100 = 16.3\%$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.5 \times 4} = 0.5 \text{ sec}$$

For 5% error, Settling time, $t_s = 3T = 3 \times 0.5 = 1.5$ sec

For 2% error, Settling time, $t_s = 4T = 4 \times 0.5 = 2$ sec

RESULT

$$\text{The time domain response, } c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$$

$$\text{(or)} \quad c(t) = 1 - \frac{2}{\sqrt{3}} e^{-2t}\left[\sin(\sqrt{12}\,t + 1.047)\right]$$

Rise time, $t_r = 0.6046$ sec

Peak time, $t_p = 0.907$ sec

% Maximum overshoot, $\%M_p = 16.3\%$

Settling time, $t_s = 1.5$ sec, for 5% error

$= 2$ sec, for 2% error

7) A unity feedback control system is characterized by the following open loop transfer function G(s) = (0.4 s +1)/s(s +0.6). Determine its transient response for unit step input and sketch the response. Evaluate the maximum overshoot and the corresponding peak time.

SOLUTION

$$\text{The closed loop transfer function, } \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$

Given that, $G(s) = (0.4s + 1)/s(s + 0.6)$

For unity feedback system, $H(s) = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{0.4\,s+1}{s(s+0.6)}}{1+\frac{0.4\,s+1}{s(s+0.6)}} = \frac{0.4\,s+1}{s\,(s+0.6)+0.4\,s+1}$$

$$= \frac{0.4\,s+1}{s^2+0.6s+0.4s+1} = \frac{0.4\,s+1}{s^2+s+1}$$

The s - domain response, $C(s) = R(s) \times \frac{0.4\,s+1}{s^2+s+1}$

For step input, R(s) = 1/s.

$$\therefore C(s) = \frac{1}{s}\,\frac{0.4\,s+1}{s^2+s+1} = \frac{0.4\,s+1}{s(s^2+s+1)}$$

By partial fraction expansion C(s) can be expressed as,

$$C(s) = \frac{0.4\,s+1}{s\,(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

The residue A is solved by multiplying C(s) by s and letting s = 0.

$$\therefore A = C(s) \times s\Big|_{s=0} = \frac{0.4\,s+1}{s^2+s+1}\Big|_{s=0} = 1$$

The residues B and C are solved by cross multiplying the following equation and equating the coefficients of like powers of s.

$$\frac{0.4\,s+1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

On cross multiplication we get,

$$0.4\,s+1 = A(s^2+s+1)+(Bs+C)\,s$$

$$0.4\,s+1 = As^2+As+A+Bs^2+Cs$$

On equating coefficients of s^2 we get, $0 = A+B$ $\therefore B = -A = -1$

On equating coefficients of s we get, $0.4 = A+C$ $\therefore C = 0.4 - A = -0.6$

$$\therefore\ C(s)=\frac{1}{s}+\frac{-s-0.6}{s^2+s+1}=\frac{1}{s}-\frac{s+0.6}{s^2+s+0.25+0.75}=\frac{1}{s}-\frac{s+0.6}{\left(s^2+2\times0.5s+0.5^2\right)+0.75}$$

$$=\frac{1}{s}-\frac{s+0.5+0.1}{(s+0.5)^2+0.75}=\frac{1}{s}-\frac{s+0.5}{(s+0.5)^2+0.75}-\frac{0.1}{\sqrt{0.75}}\ \frac{\sqrt{0.75}}{(s+0.5)^2+0.75}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

∴ The response in time domain,

$$c(t)=\mathcal{L}^{-1}\{C(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s+0.5}{(s+0.5)^2+0.75}-\frac{0.1}{\sqrt{0.75}}\frac{\sqrt{0.75}}{(s+0.5)^2+0.75}\right\}$$

$$=1-e^{-0.5t}\cos\sqrt{0.75}\,t-\frac{0.1}{\sqrt{0.75}}\ e^{-0.5t}\sin\sqrt{0.75}\,t$$

$$=1-e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t)+\cos(\sqrt{0.75}\,t)\right]$$

The transient response is the part of the output which vanishes as t tends to infinity. Here as t tends to infinity the exponential component $e^{-0.5t}$ tends to zero. Hence the transient response is given by the damped sinusoidal component.

The transient response of $c(t)=e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t)+\cos(\sqrt{0.75}\,t)\right]$

The value of ζ and ω_n can be estimated by comparing the characteristic equation of the system with standard form of second order characteristic equation.

$$\therefore\ s^2+2\zeta\omega_n s+\omega_n^2=s^2+s+1$$

On comparing we get,

$\omega_n^2=1$	$2\zeta\omega_n=1$
$\therefore\ \omega_n=1\text{ rad / sec}$	$\therefore\ \zeta=\frac{1}{2\omega_n}=\frac{1}{2}=0.5$

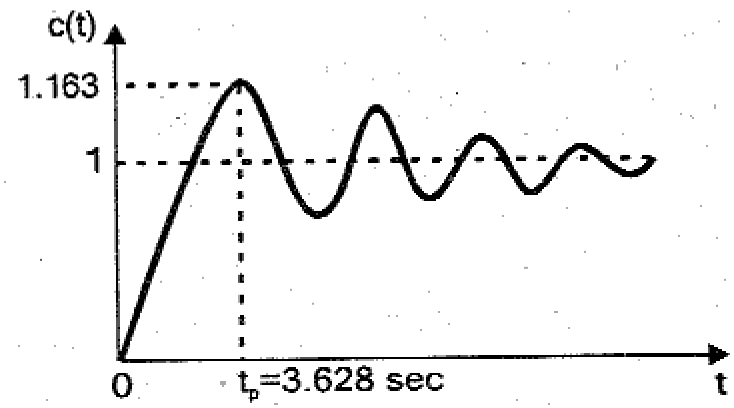
Maximum overshoot, $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} = 0.163$

% Maximum overshoot, $\%M_p = M_p \times 100 = 0.163 \times 100 = 16..3\%$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{1\times\sqrt{1-0.5^2}} = 3.628$ sec

The response of the system is underdamped and it is shown in

fig



RESULT

Transient response of the system, c(t) = $e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$

% Maximum peak overshoot, $\%M_p$ = 16.3 %

Peak time, t_p = 3.628 sec

8) A unity feedback control system has an open loop transfer function, G(s) = 10/s(s+2). Find the rise time, percentage overshoot, peak time and settling time for a step input of 12 units.

SOLUTION

Note : The formulae for rise time, percentage overshoot and peak time remains same for unit step and step input.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

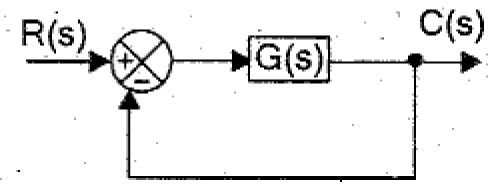


Fig 1 : Unity feedback system.

The closed loop transfer function,

Given that, G(s) = 10/s (s+2)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} = \frac{10}{s(s+2)+10} = \frac{10}{s^2+2s+10} \quad ..... (1)$$

The values of damping ratio ζ and natural frequency of oscillation ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

Standard form of Second order transfer function $\left.\right\}$ $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

On comparing equation (1) & (2) we get,

$\omega_n^2 = 10$	$2\zeta\omega_n = 2$
$\therefore\ \omega_n = \sqrt{10} = 3.162 \text{ rad / sec}$	$\therefore\ \zeta = \frac{2}{2\omega_n} = \frac{1}{3.162} = 0.316$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-0.316^2}}{0.316} = 1.249 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.162\sqrt{1-0.316^2} = 3 \text{ rad / sec}$$

$$\text{Rise time, } t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.249}{3} = 0.63 \text{ sec}$$

$$\text{Percentage overshoot, } \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.316\pi}{\sqrt{1-0.316^2}}} \times 100$$

$$= 0.3512 \times 100 = 35.12\%$$

$$\text{Peak overshoot} = \frac{35.12}{100} \times 12 \text{ units} = 4.2144 \text{ units}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.047 \text{ sec}$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.316 \times 3.162} = 1\text{ sec}$$

$\therefore$ For 5% error, Settling time, $t_s = 3T = 3$ sec

For 2% error, Settling time, $t_s = 4T = 4$ sec

RESULT

Rise time, t_r	=	0.63 sec
Percentage overshoot, $\%M_p$	=	35.12%
Peak overshoot	=	4.2144 units, (for a input of 12 units)
Peak time, t_p	=	1.047 sec
Settling time, t_s	=	3 sec for 5% error
	=	4 sec for 2% error

9) A closed loop servo is represented by the differential equation $\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64\ e$

Where c is the displacement of the output shaft, r is the displacement of the input shaft and $e = r - c$. Determine undamped natural frequency, damping ratio and percentage maximum overshoot for unit step input.

SOLUTION

The mathematical equations governing the system are,

$$\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64\ e$$

$$e = r - c$$

Put e = r - c in equation

$$\therefore \frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64(r - c)$$

Let $\mathcal{L}\{c\} = C(s)$ and $\mathcal{L}\{r\} = R(s)$

On taking Laplace transform of equation (3) we get,

$$s^2 C(s) + 8s\, C(s) = 64\,[R(s) - C(s)]$$

$$\therefore s^2 C(s) + 8s\, C(s) + 64\, C(s) = 64\, R(s)$$

$$(s^2 + 8s + 64)\, C(s) = 64\, R(s)$$

$$\therefore \frac{C(s)}{R(s)} = \frac{64}{s^2 + 8s + 64}$$

The ratio C(s)/R(s) is the closed loop transfer function of the system. On comparing the system transfer function with standard form of second order transfer function, we can estimate the values of ζ and ω_n.

$$\left.\begin{array}{l}\text{Standard form of} \\ \text{Second order transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \ldots$$

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 64 \qquad\qquad 2\zeta\omega_n = 8$$

$$\therefore \omega_n = 8 \text{ rad / sec} \qquad\qquad \zeta = \frac{8}{2\omega_n} = \frac{8}{2 \times 8} = 0.5$$

Percentage peak overshoot, $\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} \times 100 = 16.3\%$

RESULT

Undamped natural frequency of oscillation, ω_n = 8 rad/sec

Damping ratio, ζ = 0.5

Percentage peak overshoot, $\%M_p$ = 16.3%

TYPE NUMBER OF CONTROL SYSTEMS

The type number is specified for loop transfer function G(s) H(s). The number of poles of the loop transfer function lying at the origin decides the type number of the system. In general, if **N** is the number of poles at the origin then the type number is **N.**

The loop transfer function can be expressed as a ratio of two polynomials in **s.**

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,..........}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,.......} \quad .....(2.49$$

where, $z_1, z_2, z_3,$ .... are zeros of transfer function

$p_1, p_2, p_3,$ .... are poles of transfer function

K = Constant

N = Number of poles at the origin

The value of **N** in the denominator polynomial of loop transfer function shown in equation (2.49 decides the type number of the system.

If N = 0, then the system is type – 0 system

If N = 1, then the system is type – 1 system

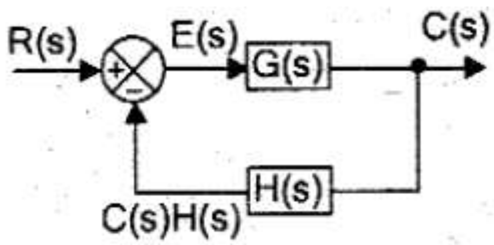
If N = 2, then the system is type – 2 system

If N = 3, then the system is type – 3 system and so on.

STEADY STATE ERROR

The steady state error is the value of error signal e(t), when **t** tends to infinity. The steady stat error is a measure of system accuracy. These errors arise from the nature of inputs, type of system an from non linearity of system components. The steady state performance of a stable control system i generally judged by its steady state error to step, ramp and parabolic inputs.

Consider a closed loop system shown in fig



Let, R(s) = Input signal

E(s) = Error signal

C(s) H(s) = Feedback signal

C(s) = Output signal or response

The error signal, $E(s) = R(s) - C(s)\,H(s)$

The output signal, $C(s) = E(s)\,G(s)$

$$E(s) = R(s) - [E(s)\,G(s)]\,H(s)$$

$$E(s) + E(s)\,G(s)\,H(s) = R(s)$$

$$E(s)\,[1 + G(s)\,H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1 + G(s)\,H(s)}$$

Let, e(t) = error signal in time domain.

$$\therefore e(t) = \mathcal{L}^{-1}\{E(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)}{1+G(s)\,H(s)}\right\}$$

Let, e_{ss} = steady state error.

The steady state error is defined as the value of e(t) when **t** tends to infinity.

$$\therefore e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\ e(t)$$

The final value theorem of Laplace transform states that,

If, $\mathbf{F(s) = \mathcal{L}\{f(t)\}}$ **then,** $\mathbf{\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s\,F(s)}$

Using final value theorem,

The steady state error, $e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\ e(t) = \underset{s\to 0}{\mathrm{Lt}}\ sE(s) = \underset{s\to 0}{\mathrm{Lt}}\ \dfrac{sR(s)}{1+G(s)\,H(s)}$

STATIC ERROR CONSTANTS

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. The value of steady state error depends on the type number and the input signal. Type-0 system will have a constant steady state error when the input is step signal. Type-1 system will have a constant steady state error when the input is ramp signal or velocity signal. Type-2 system will have a constant steady state error when the input is parabolic signal or acceleration signal.

Positional error constant, $K_p = \underset{s \to 0}{Lt}\ G(s)\,H(s)$

Velocity error constant, $K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s)$

Acceleration error constant, $K_a = \underset{s \to 0}{Lt}\ s^2 G(s)\,H(s)$

The K_p, K_v and K_a are in general called static error constants.

Steady state error when input signal is unit step signal:

Steady state error, $e_{ss} = \underset{s \to 0}{Lt}\ \dfrac{sR(s)}{1+G(s)\,H(s)}$

When the input is unit step, $R(s) = 1/s$

$$\therefore e_{ss} = \underset{s \to 0}{Lt}\ \frac{s\frac{1}{s}}{1+G(s)\,H(s)} = \underset{s \to 0}{Lt}\ \frac{1}{1+G(s)\,H(s)} = \frac{1}{1+\underset{s \to 0}{Lt}\ G(s)\,H(s)} = \frac{1}{1+K_p}$$

where, $K_p = \underset{s \to 0}{Lt}\ G(s)\,H(s)$

The constant K_p is called ***positional error constant.***

Type-0 system

$$K_p = \underset{s\to 0}{\text{Lt}}\, G(s)\,H(s) = \underset{s\to 0}{\text{Lt}}\, K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

Type-1 system

$$K_p = \underset{s\to 0}{\text{Lt}}\, G(s)\,H(s) = \underset{s\to 0}{\text{Lt}}\, K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 1 and above, for unit step input the value of K_p is infinity and so the steady state error is zero.

STEADY STATE ERROR WHEN THE INPUT IS UNIT RAMP SIGNAL

Steady state error, $e_{ss} = \underset{s\to 0}{\text{Lt}}\ \dfrac{sR(s)}{1+G(s)\,H(s)}$

When the input is unit ramp, $R(s) = \dfrac{1}{s^2}$

$$\therefore e_{ss} = \underset{s\to 0}{\text{Lt}}\ \frac{s\frac{1}{s^2}}{1+G(s)\,H(s)} = \underset{s\to 0}{\text{Lt}}\ \frac{1}{s+sG(s)\,H(s)} = \frac{1}{\underset{s\to 0}{\text{Lt}}\ sG(s)\,H(s)} = \frac{1}{K_v}$$

where $K_v = \text{Lt}\ s\,G(s)\,H(s)$

Type-0 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = 1/K_v = 1/0 = \infty$$

Hence in type-0 systems when the input is unit ramp, the steady state error is infinity.

Type-1 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = 1/K_v = \text{constant}$$

Hence in type-1 systems when the input is unit ramp there will be a constant steady state error.

Type-2 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = 1/K_v = 1/\infty = 0$$

In systems with type number 2 and above, for unit ramp input, the value of K_v is infinity so the steadv state error is zero.

STEADY STATE ERROR WHEN THE INPUT IS UNIT PARABOLIC SIGNAL

Steady state error, $e_{ss} = \underset{s\to 0}{Lt} \dfrac{sR(s)}{1+G(s)\,H(s)}$

When the input is unit parabola, $R(s) = \dfrac{1}{s^3}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt} \frac{s\frac{1}{s^3}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt} \frac{1}{s^2+s^2\,G(s)\,H(s)} = \frac{1}{\underset{s\to 0}{Lt}\; s^2G(s)\,H(s)} = \frac{1}{K_a}$$

$$\text{where, } K_a = \underset{s\to 0}{Lt}\; s^2G(s)\,H(s)$$

The constant K_a is called ***acceleration error constant.***

Type-0 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-0 systems for unit parabolic input, the steady state error is infinity.

Type-1 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity.

Type-2 system

$$K_a = \underset{s\to 0}{Lt}\ s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\ s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3.....} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \text{constant}$$

Hence in type-2 system when the input is unit parabolic signal there will be a constant steady state error.

Type-3 system

$$K_a = \underset{s\to 0}{Lt}\ s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\ s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^3\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

In systems with type number 3 and above for unit parabolic input the value of K_a is infinity and so the steady state error is zero.

TABLE-2.2 : Static Error Constant for Various Type Number of Systems

Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

TABLE-2.3 : Steady State Error for Various Types of Inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

1)

For a unity feedback control system the open loop transfer function, $G(s) = \frac{10(s+2)}{s^2(s+1)}$. Find

a) the position, velocity and acceleration error constants,

b) the steady state error when the input is R(s), where $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

SOLUTION

a) To find static error constants

For a unity feedback system , H(s)=1

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s)H(s) = \underset{s\to 0}{Lt}\, G(s) = \underset{s\to 0}{Lt}\, \frac{10(s+2)}{s^2(s+1)} = \infty$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, s\,G(s)H(s) = \underset{s\to 0}{Lt}\, s\,G(s) = \underset{s\to 0}{Lt}\, s\frac{10(s+2)}{s^2(s+1)} = \infty$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2G(s)H(s) = \underset{s\to 0}{Lt}\, s^2G(s)$

$$= \underset{s\to 0}{Lt}\, s^2\, \frac{10(s+2)}{s^2(s+1)} = \frac{10\times 2}{1} = 20$$

The error signal in s - domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$

Given that, $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$; $G(s) = \frac{10(s+2)}{s^2(s+1)}$; $H(s)=1$

$$\therefore E(s) = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{1+\frac{10(s+2)}{s^2(s+1)}} = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{\frac{s^2(s+1)+10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, s\,E(s)$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\, s\left\{\frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2s(s+1)}{s^2(s+1)+10(s+2)} + \frac{(s+1)}{3s^2(s+1)+30\,(s+2)}\right\} = 0 - 0 + \frac{1}{60}$$

$$= \frac{1}{60}$$

2)

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a nstant steady state error and calculate their values.

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$; b) $G(s) = \dfrac{10}{(s+2)(s+3)}$; c) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$

The open loop system has a pole at origin. Hence it is a type-1 system. In systems with type number-1, the velocity (ramp) input will give a constant steady state error.

The steady state error with unit velocity input, $e_{ss} = \dfrac{1}{K_v}$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, s\,G(s)\,H(s) = \underset{s\to 0}{Lt}\, s\,G(s)$

$$= \underset{s\to 0}{Lt}\, s\,\frac{20(s+2)}{s(s+1)(s+3)} = \frac{20\times 2}{1\times 3} = \frac{40}{3}$$

Steady state error, $e_{ss} = \dfrac{1}{K_v} = \dfrac{3}{40} = 0.075$

b) $G(s) = \frac{10}{(s+2)(s+3)}$

Let us assume unity feedback system, $\therefore$ H(s)=1.

The open loop system has no pole at origin. Hence it is a type-0 system. In systems with type number-0, the step input will give a constant steady state error.

The steady state error with unit step input, $e_{ss} = \frac{1}{1+K_p}$

Position error constant, $K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \frac{10}{(s+2)(s+3)} = \frac{10}{2\times 3} = \frac{5}{3}$

Steady state error, $e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\frac{5}{3}} = \frac{3}{3+5} = \frac{3}{8} = 0.375$

---

c) $G(s) = \frac{10}{s^2(s+1)(s+2)}$

Let us assume unity feedback system, $\therefore$ H(s)=1.

The open loop system has two poles at origin. Hence it is a type-2 system. In systems with type number-2, the acceleration (parabolic) input will give a constant steady state error.

The steady state error with unit acceleration input, $e_{ss} = \frac{1}{K_a}$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\ s^2\, G(s)H(s) = \underset{s\to 0}{Lt}\ s^2\, G(s) = \underset{s\to 0}{Lt}\ s^2\ \frac{10}{s^2(s+1)(s+2)} = \frac{10}{1\times 2} = 5$

Steady state error, $e_{ss} = \frac{1}{K_a} = \frac{1}{5} = 0.2$

RESULT

1. In system (a) with unit velocity input, Steady state error = 0.075
2. In system (b) with unit step input, Steady state error = 0.375
3. In system (c) with unit acceleration input, Steady state error = 0.2

3)

The open loop transfer function of a servo system with unity feedback is G(s) = 10/s(0.1s+1). Evaluate the static error constants of the system. Obtain the steady state error of the system, when subjected to an input given by the polynominal,

$r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2$.

To find static error constant

For unity feedback system, H(s) = 1.

$\therefore$ Loop transfer function, G(s) H(s) = G(s)

The static error constants are K_p, K_v and K_a.

$$\text{Position error constant, } K_p = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt} \frac{10}{s(0.1s+1)} = \infty$$

$$\text{Velocity error constant, } K_v = \underset{s\to 0}{Lt}\ sG(s) = \underset{s\to 0}{Lt}\ s\frac{10}{s(0.1s+1)} = 10$$

$$\text{Acceleration error constant, } K_a = \underset{s\to 0}{Lt}\ s^2G(s) = \underset{s\to 0}{Lt}\ s^2\frac{10}{s(0.1s+1)} = 0$$

The error signal in s - domain, $E(s)=\frac{R(s)}{1+G(s)H(s)}$

Given that, $r(t)=a_o+a_1t+\frac{a_2}{2}t^2;$ $G(s)=\frac{10}{s(0.1s+1)};$ $H(s)=1$

On taking Laplace transform of r(t) we get R(s),

$$\therefore\ R(s)=\frac{a_0}{s}+\frac{a_1}{s^2}+\frac{a_2}{2}\frac{2!}{s^3}=\frac{a_0}{s}+\frac{a_1}{s^2}+\frac{a_2}{s^3}$$

$$\therefore\ E(s)=\frac{R(s)}{1+G(s)H(s)}=\frac{\frac{a_o}{s}+\frac{a_1}{s^2}+\frac{a_2}{s^3}}{1+\frac{10}{s\,(0.1s+1)}}=\frac{\frac{a_o}{s}+\frac{a_1}{s^2}+\frac{a_2}{s^3}}{\frac{s\,(0.1s+1)+10}{s\,(0.1s+1)}}$$

$$=\frac{a_0}{s}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right]+\frac{a_1}{s^2}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right]+\frac{a_2}{s^3}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error, $e_{ss}=\underset{t\to\infty}{Lt}\ e(t)=\underset{s\to 0}{Lt}\ s\,E(s)$

$$\therefore e_{ss}=\underset{s\to 0}{Lt}\ s\left\{\frac{a_o}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]+\frac{a_1}{s^2}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]+\frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]\right\}$$

$$=\underset{s\to 0}{Lt}\left\{\frac{a_os(0.1s+1)}{s(0.1s+1)+10}+\frac{a_1(0.1s+1)}{s(0.1s+1)+10}+\frac{a_2(0.1s+1)}{s[s(0.1s+1)+10]}\right\}=0+\frac{a_1}{10}+\infty\ =\infty$$

4) Consider a unity feedback system with a closed loop transfer function $\frac{C(s)}{R(s)} = \frac{Ks+b}{s^2+as+b}$. Determine open loop transfer function G(s). Show that steady state error with unit ramp input is given by $\frac{(a-K)}{b}$.

For unity feedback system, H(s)=1

The closed loop transfer function, $M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G(s)}$

$$\therefore \frac{G(s)}{1+G(s)} = M(s)$$

On cross multiplication of the above equation we get,

$$G(s) = M(s)[1+G(s)] = M(s) + M(s)G(s)$$

$$\therefore G(s) - M(s)G(s) = M(s) \Rightarrow G(s)[1 - M(s)] = M(s) \Rightarrow M(s) = \frac{Ks+b}{s^2+as+b}$$

∴ Open loop transfer function,

$$G(s) = \frac{M(s)}{1-M(s)} = \frac{\frac{Ks+b}{s^2+as+b}}{1-\frac{Ks+b}{s^2+as+b}} = \frac{Ks+b}{(s^2+as+b)-(Ks+b)}$$

$$= \frac{Ks+b}{s^2+as+b-Ks-b} = \frac{Ks+b}{s^2+(a-k)s} = \frac{Ks+b}{s[s+(a.K)]}$$

Velocity error constant, $K_v = \underset{s \to 0}{Lt}\ s\,G(s)H(s) = \underset{s \to 0}{Lt}\ s\,G(s) = \underset{s \to 0}{Lt}\ s\frac{Ks+b}{s[s+(a-K)]} = \frac{b}{a-K}$

With velocity input, Steady state error, $e_{ss} = \frac{1}{K_v} = \frac{a-K}{b}$

RESULT

Open loop transfer function, $G(s) = \frac{Ks+b}{s\,[s+(a-K)]}$

With veloctiy input, Steady state error, $e_{ss} = \frac{a-K}{b}$